

# Magnetic Moment of Electrons near Cosmic Strings

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## Abstract

We study the effect of background geometry generated by a thin cosmic string on the anomalous magnetic moment of the electron. We find that the magnitude of the quantum correction to the magnetic moment depends on the distance from the cosmic string as well as on the deficit angle.

## 1 Introduction

Topological defects will arise in some of the models with spontaneous symmetry breakdown in field theory. Stringlike objects, which are generated by the breakdown of U(1) symmetry, have the possibility of providing the seeds for galaxies in the early universe.[1, 2, 3] The cosmic strings are expected to have large mass density and very thin width. The space-time geometry around an infinitely stretching straight string has a peculiar property. Suppose that the idealized cosmic string lies with energy density (per unit length)  $\mu$  on the  $z$  axis. The metric of the space-time is found to be [1]

$$ds^2 = -dt^2 + dr^2 + \frac{r^2}{\nu^2} d\theta^2 + dz^2, \quad (1)$$

where  $\nu^{-1} = 1 - 4G\mu < 1$ , and  $r$  and  $\theta$  are the polar coordinates of  $x$ - $y$  plane. If we use a new coordinate,

$$\theta' \equiv \frac{\theta}{\nu}, \quad (2)$$

the metric reduces to the flat space-time except for the deficit in the azimuthal angle because  $\theta'$  takes a value from zero to  $2\pi/\nu$ . Thus one can say that the space has a conical singularity at the location of the idealized cosmic string.

Many authors have studied the quantum field theory around the conical space.[4] In such studies, the explicit calculation of quantum effects is carried out owing to the local flatness of the space-time. They have the same origin as the Casimir force [5] between two conducting plates.

The quantum electromagnetics near the conducting plates have also been explored by many authors, including Barton et al. (see Ref. [6] and references therein). The measurement of the  $g$  factor has been planned, though there is difficulty in obtaining sufficient precision for the apparatus.

The cosmic strings, if associated with the GUT scale, are so thin that we can regard them as ideal one-dimensional objects. Therefore we can investigate quantum electromagnetism around cosmic strings in a similar way. The analysis may reveal various types of quantum effects near cosmic strings, which have not yet been calculated.

In the present paper, we calculate the anomalous magnetic moment of the electron near a straight cosmic string with infinitesimal width. In other words, we study quantum mechanics of an electron in the background space-time expressed by the metric (1). For simplicity we consider the external magnetic field parallel to the cosmic string which lies on the  $z$  axis.

We must consider Ref. [6] when estimating the contribution to the magnetic moment correction, though the manipulation related to the regularization is quite different. The organization of this paper will inevitably resemble the intensive work Ref. [6].

Section 2 shows the preparation for the quantum-mechanical calculations. The mode expansion of the electromagnetic field in the conical spacetime is explicitly given. The renormalized value for the mode summation is calculated. We show the Hamiltonian in the nonrelativistic limit. In Sec. 3, we consider the second order perturbation to the Hamiltonian for a “fixed electron” near a cosmic string, which comes from the direct coupling term between the magnetic moment of the electron and the magnetic field. Similar calculations of the other contributions to the anomalous magnetic moment of the electron near the cosmic string are performed in Sec. 4. Section 5 is devoted to a summary and discussion.

## 2 Preliminary: Mode Expansion of the Maxwell Field in the Conical Space and the Hamiltonian

We take the Coulomb gauge on the vector potential. Then  $\mathbf{E} = -\dot{\mathbf{A}}$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ .

The normal mode expansion in the conical spacetime (1) is written in the form

$$A_i \approx \int_0^\infty dl l \int_{-\infty}^\infty dk \frac{1}{\sqrt{2\omega}} \sum_n \sum_{s=1,2} a^s A_i^s(r, \theta) e^{ikz - i\omega t} + h.c., \quad (3)$$

where the mode functions  $A_i^s$  are given by the sets

$$A_z^1 = \left\{ i\frac{1}{\omega}J_{\nu n}(lr)\sin n\theta, \quad i\frac{1}{\omega}J_{\nu n}(lr)\cos n\theta \right\}, \quad (4)$$

$$A_r^1 = \left\{ \frac{k}{2\omega}[J_{\nu n+1}(lr) - J_{\nu n-1}(lr)]\sin n\theta, \right. \\ \left. \frac{k}{2\omega}[J_{\nu n+1}(lr) - J_{\nu n-1}(lr)]\cos n\theta \right\}, \quad (5)$$

$$A_\theta^1 = \left\{ -\frac{kr}{2\nu\omega}[J_{\nu n+1}(lr) + J_{\nu n-1}(lr)]\cos n\theta, \right. \\ \left. \frac{kr}{2\nu\omega}[J_{\nu n+1}(lr) + J_{\nu n-1}(lr)]\sin n\theta \right\}, \quad (6)$$

$$A_z^2 = 0, \quad (7)$$

$$A_r^2 = \left\{ \frac{1}{2}[J_{\nu n+1}(lr) + J_{\nu n-1}(lr)]\sin n\theta, \right. \\ \left. \frac{1}{2}[J_{\nu n+1}(lr) + J_{\nu n-1}(lr)]\cos n\theta \right\}, \quad (8)$$

$$A_\theta^2 = \left\{ -\frac{1}{2}[J_{\nu n+1}(lr) - J_{\nu n-1}(lr)]\cos n\theta, \right. \\ \left. \frac{1}{2}[J_{\nu n+1}(lr) - J_{\nu n-1}(lr)]\sin n\theta \right\}, \quad (9)$$

$J_m(z)$  being the Bessel function and  $\omega = \sqrt{k^2 + l^2}$ .

We briefly denote (3) as

$$A_i = \sum_{\lambda} a_{\lambda} \bar{A}_{i\lambda}(r, \theta, z) e^{-i\omega t} + h.c., \quad (10)$$

where the single label  $\lambda$  includes  $k, l, n$  and  $s$ , and the  $z$  dependence and the appropriate normalization are included in  $\bar{A}_{i\lambda}$ . Accordingly, we can write

$$\mathbf{B} = \nabla \times \mathbf{A} = \sum_{\lambda} a_{\lambda} \bar{\mathbf{B}}_{\lambda}(r, \theta) e^{-i\omega t} + h.c., \quad (11)$$

etc. Upon the second quantization,  $a$  and  $a^\dagger$  are replaced by the annihilation and creation operators. The calculation for perturbative corrections can be written in simple forms by using the notation.

The notation can be used for the “inner product” of the fields, which is substantial in our calculation. We can express the vacuum expectation value of

the bilinear of quantum fields by using the mode summation.[7] For example, the unrenormalized expression for the expectation value for the product of the vector field is written by

$$\langle A_z A^z \rangle = \sum_{\lambda} |\bar{A}_{z\lambda}|^2 = \frac{\nu}{\pi} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{l^2}{\omega^2} J_{\nu n}^2(lr). \quad (12)$$

Similarly, one can obtain the expression

$$\begin{aligned} \langle A_r A^r + A_\theta A^\theta \rangle &= \sum_{\lambda} (\bar{A}_{r\lambda} \bar{A}_\lambda^{r*} + \bar{A}_{\theta\lambda} \bar{A}_\lambda^{\theta*}) \\ &= \frac{\nu}{\pi} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{k^2 + \omega^2}{2\omega^2} [J_{\nu n+1}^2(lr) + J_{\nu n-1}^2(lr)]. \end{aligned} \quad (13)$$

The regularization for these quantities is rather simple in our case. We can use Smith's method of regularization (see his paper in Ref. [4]). His method is essentially equal to the point-splitting; one can consider the small separation limit after replacing the arguments in the two Bessel functions with  $lr$  and  $lr'$ . An example of such calculation is given in the Appendix. Here we write only the renormalized results:

$$\left\{ \sum_{\lambda} |\bar{A}_{z\lambda}|^2 \right\}^{(R)} = \frac{\nu^2 - 1}{48\pi^2 r^2}, \quad (14)$$

$$\left\{ \sum_{\lambda} (\bar{A}_{r\lambda} \bar{A}_\lambda^{r*} + \bar{A}_{\theta\lambda} \bar{A}_\lambda^{\theta*}) \right\}^{(R)} = \frac{\nu^2 - 1}{48\pi^2 r^2}. \quad (15)$$

We can also regularize the expression which appears in the perturbative calculation. For later use, we calculate the quantities

$$\sum_{\lambda} \frac{1}{\omega_\lambda^2} |\bar{B}_{z\lambda}|^2 = \frac{\nu}{\pi} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{l^2}{\omega^2} J_{\nu n}^2(lr), \quad (16)$$

$$\begin{aligned} \sum_{\lambda} \frac{1}{\omega_\lambda^2} (\bar{B}_{r\lambda} \bar{B}_\lambda^{r*} + \bar{B}_{\theta\lambda} \bar{B}_\lambda^{\theta*}) \\ = \frac{\nu}{\pi} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{k^2 + \omega^2}{2\omega^2} [J_{\nu n+1}^2(lr) + J_{\nu n-1}^2(lr)]. \end{aligned} \quad (17)$$

These mathematical expressions coincide with the previous ones for vector fields. Thus we immediately get the renormalized values:

$$\left\{ \sum_{\lambda} \frac{1}{\omega_\lambda^2} |\bar{B}_{z\lambda}|^2 \right\}^{(R)} = \frac{\nu^2 - 1}{48\pi^2 r^2}, \quad (18)$$

$$\left\{ \sum_{\lambda} \frac{1}{\omega_{\lambda}^2} (\bar{B}_{r\lambda} \bar{B}_{\lambda}^{r*} + \bar{B}_{\theta\lambda} \bar{B}_{\lambda}^{\theta*}) \right\}^{(R)} = \frac{\nu^2 - 1}{48\pi^2 r^2}. \quad (19)$$

Now we turn to the Hamiltonian. Note that it takes the same form in the empty space if we use the coordinate  $\theta'$  [see (2)]. Consequently, we have only to take the effect of the wedge angle into the mode sum calculations.

For the external (classical) magnetic field, we consider a uniform magnetic field parallel to the cosmic string. We take the  $z$  axis in the direction

$$\mathbf{B}_0 = B_0 \hat{z}. \quad (20)$$

Hereafter we separate the contribution of the external field from the vector potential:  $\mathbf{A} \rightarrow \mathbf{A} + \mathbf{A}_0$ , where  $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ .

The mass and charge of the electron are denoted by  $m$  and  $e$ , respectively.

The Hamiltonian in the nonrelativistic limit is given by [6]

$$\begin{aligned} H &= H_{rad} + H_0 - \frac{e}{2m} \sigma \cdot \mathbf{B} - \frac{e}{m} \mathbf{A} \cdot \pi_0 + \frac{e}{8m^2} \sigma \cdot (\pi_0 \times \mathbf{E} - \mathbf{E} \times \pi_0) \\ &\quad + \frac{e^2}{2m} \mathbf{A}^2 + \frac{e^3}{4m^3} \mathbf{A}^2 \sigma \cdot \mathbf{B}_0 + \dots \\ &= H_{rad} + H_0 + H_M + H_E + H_{SO} + H_2 + H_3 + \dots, \end{aligned} \quad (21)$$

where  $H_{rad}$  is the Hamiltonian for the Maxwell fields and

$$H_0 = \frac{1}{2m} (\pi_0^2 - e \sigma \cdot \mathbf{B}_0)^2, \quad (22)$$

with

$$\pi_0 = -i\nabla - e\mathbf{A}_0. \quad (23)$$

Here we omit the image energy (proportional to  $e^2/r$ ), which is irrelevant to the present calculation for the anomalous magnetic moment.

We will calculate the correction to the magnetic moment in the case of  $B_0 r \ll 1$ , for we wish to know the leading order. Then we do not have to worry about the other higher order terms in the expansion of the Hamiltonian.

### 3 Perturbative Calculation for a “Fixed Electron”

In this section, we treat an electron of fixed position, which is abbreviated to a “fixed electron.” [5] We adopt the Hamiltonian in the limit of infinite mass for the electron here. The treatment of this quantum-mechanical correction to the magnetic moment is simple but will be found to be short. The other contribution from quantum “induction” of the electromagnetic field will be treated in Sec. 4.

The spin of the electron is directly coupled to the external magnetic field. The interaction Hamiltonian we examine in this section is simply written as

$$\begin{aligned} H_I &= -\mu \cdot \mathbf{B}_0 - \mu \cdot \mathbf{B} + H_{rad} \\ &= -\frac{e}{2m} \sigma \cdot \mathbf{B}_0 - \frac{e}{2m} \sigma \cdot \sum_{\lambda} (a_{\lambda} \bar{B}_{\lambda} + a_{\lambda}^{\dagger} \bar{B}_{\lambda}^*) + \sum_{\lambda} \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda}, \end{aligned} \quad (24)$$

where  $\sigma/2$  is the spin of the electron ( $J = 1/2$ ), and  $\mu$  is the magnetic moment,  $\mu = (e/m)(\sigma/2)$ . By a “fixed electron”, we mean that only the interaction of the magnetic moment and the magnetic field is taken into consideration.

The unrenormalized (second order) perturbation is given by [5]

$$H_{eff} = - \sum_{\lambda} \sum_{M'=\pm 1/2} \left| \left\langle M' \left| \frac{e}{2m} \sigma \cdot \bar{\mathbf{B}}_{\lambda} \right| M \right\rangle \right|^2 \frac{P}{\omega_{\lambda} + \left| \frac{e}{m} \right| B_0 (M' - M)}, \quad (25)$$

where  $P$  means the principal value. Using the notation introduced in Sec. 2, one can reduce (25) to

$$H_{eff} = - \frac{e^2}{m^2} \sum_{\lambda} \left\{ \frac{M^2}{\omega_{\lambda}} |\bar{\mathbf{B}}_{z\lambda}|^2 + \frac{1}{4} \left[ \frac{\frac{1}{2} - M}{\omega_{\lambda} + \left| \frac{e}{m} \right| B_0} + \frac{(\frac{1}{2} + M) P}{\omega_{\lambda} - \left| \frac{e}{m} \right| B_0} \right] (\bar{B}_{r\lambda} \bar{B}_{\lambda}^{r*} + \bar{B}_{\theta\lambda} \bar{B}_{\lambda}^{\theta*}) \right\} \quad (26)$$

The leading contribution to the anomalous magnetic moment is a linear part in  $MB_0$  if we expand (26) in the weak field limit. We find that

$$-\delta\mu \cdot \mathbf{B}_0 = \frac{e^3}{4m^3} \sum_{\lambda} \frac{2MB_0}{\omega_{\lambda}^2} (\bar{B}_{r\lambda} \bar{B}_{\lambda}^{r*} + \bar{B}_{\theta\lambda} \bar{B}_{\lambda}^{\theta*}). \quad (27)$$

Using the renormalized value (19), we obtain

$$\left( \frac{\delta\mu}{\mu} \right)_{fixed} = - \frac{(\nu^2 - 1)e^2}{96\pi^2 m^2 r^2}. \quad (28)$$

The correction to the magnetic moment of the same order of magnitude comes from the other term in the Hamiltonian (21). We study them in the next section.

## 4 More Perturbative Corrections

In this section, we consider other contributions to the anomalous magnetic moment of the electron near a cosmic string. By naive consideration, the result of the previous section is satisfactory in the nonrelativistic system. It was pointed out [5] that the other terms in the Hamiltonian, which are derived from the relativistic theory, bring about additional contributions to the anomalous magnetic moment of the same order as the previous result. They can be interpreted as the effect of quantum “electromagnetic induction.”

The first contribution we consider is the first order perturbation due to  $H_3$  in (21). The effective Hamiltonian which comes from this perturbation is given by

$$H_{eff}(3) = \frac{e^3}{4m^3} \left( \sum_{\lambda} |\bar{\mathbf{A}}_{\lambda}|^2 \right) \sigma \cdot \mathbf{B}_0. \quad (29)$$

Using the renormalized values (14) and (15), we get

$$H_{eff}^{(R)}(3) = \frac{(\nu^2 - 1)e^3}{96\pi^2 m^3 r^2} \sigma \cdot \mathbf{B}_0. \quad (30)$$

The next contribution originates from the second order perturbation due to  $H_E$  and  $H_{SO}$ . After some manipulation, we get the effective Hamiltonian in the presence of the parallel magnetic field with the cosmic string:

$$H_{eff}(E \cdot SO) = \frac{e^3}{4m^3} \left[ \sum_{\lambda} (\bar{A}_{r\lambda} \bar{A}_{\lambda}^{r*} + \bar{A}_{\theta\lambda} \bar{A}_{\lambda}^{\theta*}) \right] \sigma_z \cdot B_0. \quad (31)$$

Applying the result (15), we obtain

$$H_{eff}^{(R)}(E \cdot SO) = \frac{(\nu^2 - 1)e^3}{192\pi^2 m^3 r^2} \sigma_z \cdot B_0. \quad (32)$$

Finally, we consider the second order perturbation due to  $H_E$  and  $H_M$  in (21). After lengthy calculation, we find that the effective Hamiltonian includes the term

$$-\frac{e^3}{2m^3} \left[ \sum_{\lambda} \frac{1}{\omega_{\lambda}^2} |\bar{B}_{\lambda}^z|^2 \right] \sigma_z \cdot B_0 \quad (33)$$

if the external magnetic field is parallel to the  $z$  axis, on which the cosmic string lies. The substitution of the renormalized value (18) yields

$$-\frac{(\nu^2 - 1)e^3}{96\pi^2 m^3 r^2} \sigma_z \cdot B_0. \quad (34)$$

The three additional contributions are of the same order as the result obtained in the previous section.

Summing over all the contributions to the magnetic moment (28), (30), (32) and (34), we find that the correction to the magnetic moment is given by

$$\frac{\delta\mu}{\mu} = -\frac{(\nu^2 - 1)e^2}{48\pi^2 m^2 r^2}. \quad (35)$$

## 5 Summary and Discussion

According to the result, the actual order of the magnitude is given by [5]

$$\frac{\delta\mu}{\mu} \approx \frac{(\nu^2 - 1)e^2}{m^2 r^2} = 1.09 \times 10^{-23} \times (\nu^2 - 1) \times (r/\text{cm})^{-2}. \quad (36)$$

Since the width of cosmic strings associated with the GUT scale is about  $10^{-28}$  cm and  $\nu - 1 \approx 10^{-6}$ , it is possible for electrons to approach so closely that the correction to the anomalous magnetic moment becomes large. Thus, if one can prepare a static cosmic string in his laboratory, he could measure the shifted value for the anomalous magnetic moment of electrons.

In astronomical observation, however, we do not expect to find any effect even if cosmic strings exist in our universe, because:

(1) The strong magnetic field and the high density ionized region can hardly overlap each other in the galactic space. [There is a little hope in the case where the cosmic string is located in the vicinity of a neutron star (with an accompanied star).]

(2) Cosmic strings are expected to be moving near the speed of light, so that other effects may be quite large and eliminate the effect considered in this paper. Dynamical effects on classical and quantum electromagnetism must be studied.

The calculation of the cyclotron frequency shift near a cosmic string is more difficult in our case than that near conducting plates.[6] This task is left for future works.

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## Appendix

We give an illustrative calculation for the regularization of the mode summation. We consider here the expression

$$\sum_{\lambda} |\bar{A}_{z\lambda}|^2 = \frac{\nu}{\pi} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{l^2}{\omega^2} J_{\nu n}^2(lr). \quad (37)$$

We regard this as the following quantity in the limit of  $\zeta \rightarrow 0$  and  $r' \rightarrow r$ :

$$\frac{\nu}{\pi} \frac{1}{2} \sum_n \int_0^\infty dl l \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{2\omega} \frac{l^2}{\omega^2} J_{\nu n}(lr) J_{\nu n}(lr') e^{ik\zeta}, \quad (38)$$

where  $\zeta$  is the difference of the  $z$  coordinates of two separated points.

By using the formula on the modified Bessel functions, which can be found in Smith's paper (Ref. [3]), we reduce (38) to

$$\frac{1}{4\pi^2 r r'} \frac{\nu}{\sinh \frac{\nu u}{2}} \frac{\cosh u - 1}{\sinh^2 u} \left( \frac{\cosh u}{\sinh u} \cosh \frac{\nu u}{2} + \frac{\nu}{2 \sinh \frac{\nu u}{2}} \right), \quad (39)$$

where

$$\cosh u \equiv \frac{r^2 + r'^2 + \zeta^2}{2rr'}. \quad (40)$$

For small  $u$ , this can be expanded as

$$\frac{1}{2\pi^2 r r'} \frac{1}{u^2} \left( 1 + \frac{1}{12} u^2 + \frac{\nu^2 u^2}{24} + \dots \right). \quad (41)$$



We can subtract the result for the empty space — which can be obtained by setting  $\nu = 1$  — from the expression (41). After the subtraction, we can take the limit  $u \rightarrow 0$  and  $r' \rightarrow r$ . This operation leads to

$$\left\{ \sum_{\lambda} |\bar{A}_{z\lambda}|^2 \right\}^{(R)} = \frac{\nu^2 - 1}{48\pi^2 r^2}. \quad (42)$$

We can get the normalized quantities of the other types by similar methods.

## References

- [1] A. Vilenkin, Phys. Rep. **121** (1985) 263.
- [2] *Cosmic Strings: The Current Status*, eds. F. S. Accetta and L. M. Krauss (World Scientific, Singapore, 1988).
- [3] *The Formation and Evolution of Cosmic Strings*, eds. G. Gibbons, S. Hawking and T. Vachaspati (Cambridge University Press, Cambridge, 1989).
- [4] V. P. Frolov and E. M. Serebriany, Phys. Rev. **D35** (1987) 3779; B. Linet, Phys. Rev. **D35** (1987) 536; J. S. Dowker, Phys. Rev. **D36** (1987) 3742; J. S. Dowker, in Ref. 3, p. 251; A. G. Smith, in Ref. 3, p. 263; A. Sarmiento and S. Hacyan, Phys. Rev. **D38** (1988) 1331; I. H. Russell and D. J. Toms, Class. Quantum Grav. **6** (1989) 1343; D. Harari and V. Skarzhinsky, Phys. Lett. **B240** (1990) 322; J. Audretsch and A. Economou, Phys. Rev. **D44** (1991) 980; S. Hirenzaki and K. Shiraishi, Class. Quantum Grav. **9** (1992) 2277.
- [5] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wet. **51** (1948) 793; G. Plunien, B. Müller and W. Greiner, Phys. Rep. **134** (1986) 87; V. M. Mostepanenko and N. N. Trunov, Sov. Phys. Usp. **31** (1988) 965.
- [6] G. Barton and N. S. J. Fawcett, Phys. Rep. **170** (1988) 1.
- [7] K. Shiraishi, J. Korean Phys. Soc. **25** (1992) 192.